

Generalizations of $Lie_{\mathbb{R}} \xrightarrow[\cong]{\cong} Moduli$

Outlook on much more general statement

Proof of above \rightarrow purely categorical part

\rightarrow certain statements specific to Com-Lie case

Today: sketch of general statement + associative case.

Example: $A \in \text{cdga}_{\mathbb{R}}^{\leq 0}$

Def'n: M a projective A -module of rank n if

(1) $\pi_0 M$ is proj. $\pi_0 A$ -module of rank n

(2) M is flat over A , i.e. $\forall k,$

$$\text{Tor}_0^{\pi_0 A}(\pi_k A, \pi_0 M) \cong \pi_k M \text{ is isom.}$$

$X(A) \subseteq \text{Mod}_A \leftarrow \infty\text{-category of } A\text{-modules}$

\uparrow objects = proj. rk n A -modules

mor = equivalences

} (essentially small)
Kan complex

$$\rightsquigarrow X: \text{cdga}_{\mathbb{R}}^{\leq 0} \longrightarrow \mathcal{S}$$

Pick $\eta \in X(\mathbb{R}) \iff V$ n -dim'l v.sp./ \mathbb{R}

$$X^\wedge(A) = X(A) \times_{X(\mathbb{R})} \{\eta\} = \text{classifying space of pairs } (M, \alpha):$$

$$X^\wedge: \text{CAlg}_{\mathbb{R}}^{\text{sm}} \longrightarrow \mathcal{S}$$

M proj of rk n ,
 $\alpha: \mathbb{R} \otimes_A M \cong V$

is a formal moduli problem.

Max-Planck-Institut für Mathematik

Bonn



$$\Omega^{\infty} T_X = X^{\wedge} (k[E]/(E^2)) \simeq BC$$

$$1 + E \cdot \text{End}(V) \simeq A = \text{Automorphisms of } k[E]/E^2 \otimes V$$

$$\equiv \text{id}_V \text{ mod } E$$

$$\downarrow \\ 1_V + E\alpha, \alpha \in \text{End}(V)$$

$$\Rightarrow \text{homology of } \mathfrak{g} \simeq \begin{cases} \cdot \text{End}(V) & \text{deg } 0 = * \\ 0 & * > 0 \end{cases}$$

$$H_*(\mathfrak{g})$$

* < 0 : $X^{\wedge} (k \oplus k[d])$ ← spectrum of spaces T_X

$$\left\{ \frac{k[x_{-d}]}{(x_{-d})^2} \right\} x_{-d} \text{ deg } -d$$

↖ spectrum of k-modules

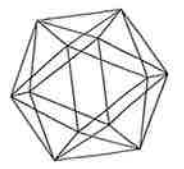
Fact: any basis of V can be lifted to one for any $k[x_{-d}]/(x_{-d})^2$ -module M
 $\Rightarrow M \simeq V \otimes k[x_{-d}]/(x_{-d})^2$ deforming V .

$\Rightarrow X^{\wedge}(\dots)$ has only one isom. class and is connected as a space.

$$\Rightarrow H_*(\mathfrak{g}) = 0 \text{ for } * < 0.$$

$\Rightarrow \mathfrak{g} \stackrel{\text{q.iso}}{\simeq} \mathfrak{g}$ ordinary Lie alg. / k

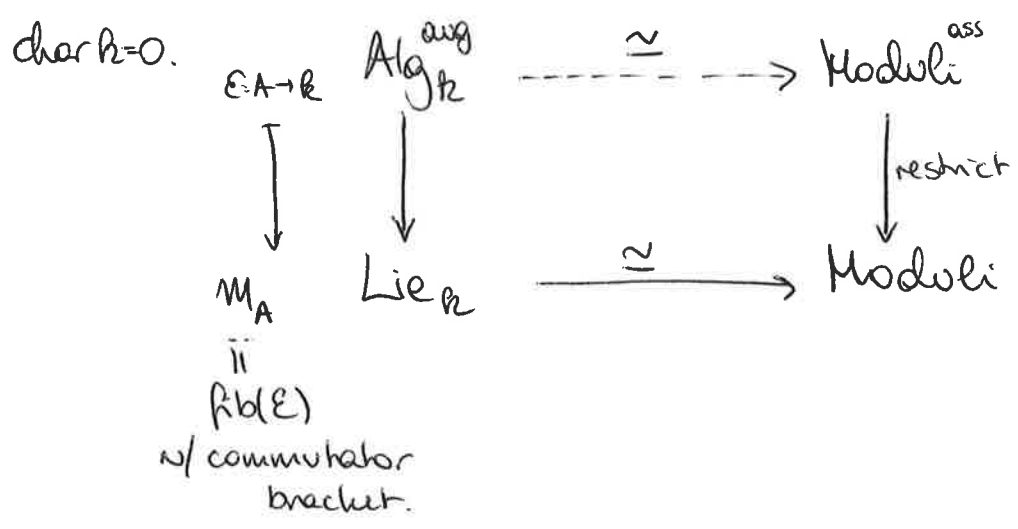
Claim: $\mathfrak{g} \simeq \text{End}(V)$ as Lie alg.



Q: What extra property does this have?

A: It's the Lie algebra coming from an augmented associative algebra using commutator bracket.

This will be a general pattern!



Q: Can we generalize even more?

- Which parts of the proof work in general, which ones are specific to the context we are in?

Lurie proves a very, very general version of which the above are special examples!

Thm (sketch): Let \mathcal{A} be an ∞ -category of some algebraic objects, let $\mathcal{A}^{\text{art}} \subseteq \mathcal{A}$ be a full subcategory of "Artinian"/"small" objects, let $\text{Moduli}^{\mathcal{A}} \subset \text{Fun}(\mathcal{A}^{\text{art}}, \mathcal{S})$ be the ∞ -cat of $X: \mathcal{A}^{\text{art}} \rightarrow \mathcal{S}$ satisfying a certain gluing condition.

$\mathcal{B}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$
is a "deformation theory"

Then there is an equivalence of ∞ -categories of the form $\text{Moduli}^{\mathcal{A}} \simeq \mathcal{B}$

Let \mathcal{B} be a certain ∞ -cat. of algebraic objects governing deformations of \mathcal{A} -obj.



Reformulation of X^1 :

If $A \in \text{CAlg}_k^{\text{sm}}$, M connective A -module (i.e. $M^i = 0 \text{ } i > 0$)
 st. $M \otimes_A k \cong V$

$\implies M$ is projective of rank n .

$\implies X^1(A) = \text{LMod}_A^{\text{conn}} \times \{V\}$
 $\text{LMod}_k^{\text{conn}}$

does not use commutativity of A !

Can

Extend:

$X^1_+ : \text{Alg}_k^{\text{art}} \longrightarrow \mathcal{S}$

\nearrow
 ∞ -category of "Artinian" associative dg algebras
 "small" " E_1 " k

Question? Is this reflected in the ~~target~~ Lie algebra? $\mathfrak{g} = \mathbb{F}(X)$
~~complex~~ / associated

Computation of T_{X^1} tangent complex:

~~target~~ $X^1(k[E]/(E^2))$ gpd of order 1 deformations of the v.sp. V :

projective $k[E]/(E^2)$ -modules M
 w/ iso $M/\epsilon M \cong V$.

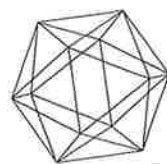
Fact: any basis for V can be lifted to one of M .

$\implies M \cong k[E]/(E^2) \otimes_k V$

morphisms in the gpd:

autom of $k[E]/(E^2) \otimes_k V$

which are identity modulo ϵ . $\stackrel{\text{uniquely}}{=} 1_V + \epsilon \cdot \alpha \quad \alpha \in \text{End}(V)$



General setup:

- A presentable ω -category ($\Rightarrow \exists$ final obj.)
- $Sp(A) =: \omega$ -cat of spectrum objects

in these lectures so far:

$A = \text{Cat}_{\mathbb{R}}^{\text{alg}} = \text{Cat}_{\mathbb{R}}^{\text{gr}}$

$\text{Mod}_{\mathbb{R}} \ni \mathbb{R}$

$Sp(A) \ni \{ \mathbb{R} \oplus \mathbb{R}[d] \}_{d \in \mathbb{N}}$
 $\cong \mathbb{R}[X_{-d}] / (X_{-d})^2$

was a spectrum E

$F: S_{+}^{\text{fin}} \rightarrow A$ • reduced + excisive

~~Remember~~

Think: $Sp(A)$ is homot of

$\dots \rightarrow A_* \xrightarrow{\Omega} A_*$
 \uparrow
 ω -cat of pt'd obj in A

$\Omega^{\infty-n} E = \mathbb{R} \oplus \mathbb{R}[n]$

$\Omega^{\infty-n}: Sp(A) \rightarrow A, \Omega^{\infty-n}(x) = \Omega^m x(S^{n+m})$ for any $m \geq -n$

Definition: A deformation context is a pair $(A, \{E_{\alpha}\}_{\alpha \in T})$

where • A is a presentable ω -cat

• $\{E_{\alpha}\}_{\alpha \in T}$ is a set of objects in $Sp(A)$

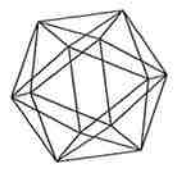
Ex: $(\text{Cat}_{\mathbb{R}}^{\text{alg}}, \{ \mathbb{R} \oplus \mathbb{R}[d] \}_{d \in \mathbb{N}})$
 $\cong E$

Definition: $\phi: A' \rightarrow A$ in A is elementary if $\exists \alpha \in T$, integer $n > 0$,

$$\begin{array}{ccc} A' & \longrightarrow & * \\ \phi \downarrow & \dashv & \downarrow \phi_0 \\ A & \longrightarrow & \Omega^{\infty-n} E_{\alpha} \end{array}$$

$$\begin{array}{ccc} A' & \longrightarrow & \mathbb{R} \\ \downarrow \dashv & & \downarrow \\ A & \longrightarrow & \mathbb{R} \oplus \mathbb{R}[n] \end{array}$$

$\phi_0 \leftrightarrow$ image of E_{α} under $Sp(A) \xrightarrow{\Omega^{\infty-n}} A$



Def'n: Let $(A, \{E_\alpha\}_{\text{det}})$ def context.

$\phi: A' \rightarrow A$ is small if it is finite composition of elem.

$$A' \simeq A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n \simeq A$$

A is Artinian (small) if $A \rightarrow *$ is small
(det. up to hom.)

$A^{\text{art}} \subset A$ full subcategory of Artinian obj.

$\text{CA}lge_r \subset \text{CA}lge_r$

Example: $\Omega^{\infty-n} E_\alpha$ is small:

$\mathbb{R} \oplus \mathbb{R}[n]$ is small

$$\begin{array}{ccc} \Omega^{\infty-n} E_\alpha & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \Omega^{\infty-n-1} E_\alpha \end{array}$$

Prop Def'n A formal moduli problem for $(A, \{E_\alpha\}_{\text{det}})$ is a

functor $X: A^{\text{art}} \rightarrow \mathcal{S}$ s.t.

(1) $X(*)$ ~~is~~ is contractible

(2) If $\sigma: \begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & \lrcorner & \downarrow \phi \\ A & \longrightarrow & B \end{array}$ in A^{art} and ϕ is small

then $\begin{array}{ccc} X(A') & \longrightarrow & X(B') \\ \downarrow & \lrcorner & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$ in \mathcal{S} .

Moduli $A \subset \text{Fun}(A^{\text{art}}, \mathcal{S})$ spanned by FMP's.

... ∞ -cat of FMP's.

Remark: Have similar proposition to previously which allows to test only for

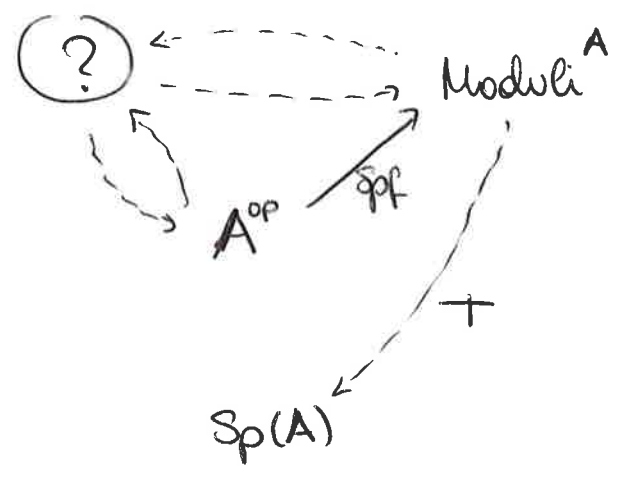
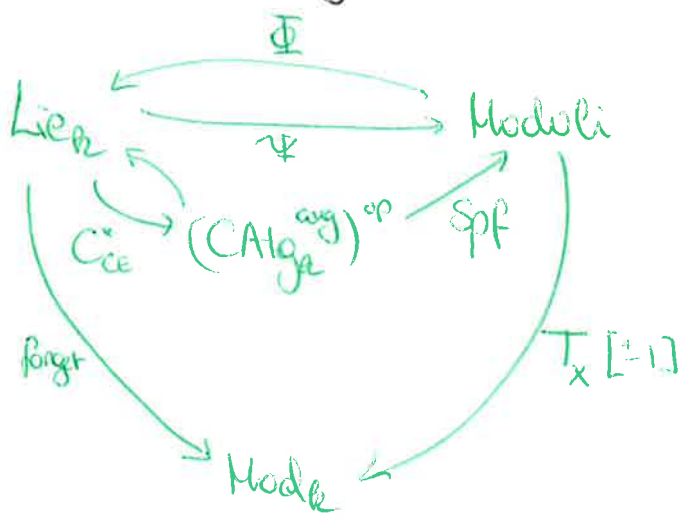
- ϕ is elementary
- or even - $\phi : * \rightarrow \Omega^{\infty-n} E_\alpha$

formal spectrum $A \in \mathcal{A}$. $\text{Spf}(A) : A^{\text{art}} \rightarrow \mathcal{S} \in \text{FMP}$

$$\text{Spf}(A)(B) = \text{Map}_A(A, B)$$

$$\rightsquigarrow \text{Spf} : \mathcal{A}^{\text{op}} \rightarrow \text{Moduli}^{\mathcal{A}}$$

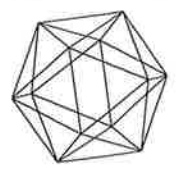
Recall our diagram:



Tangent space of $\Upsilon : A^{\text{art}} \rightarrow \mathcal{S}$ FMP at $\alpha \in T$ is $\Upsilon(\Omega^\infty E_\alpha)$

... has structure of a spectrum (of spaces):

$$S_{\text{fin}} \xrightarrow{E_\alpha} A^{\text{art}} \xrightarrow{\Upsilon} \mathcal{S} \quad \text{"tangent complex to } \Upsilon \text{ at } \alpha \text{"}$$



We get similar proposition to previously, in this abstract setting

Prop: Let $u: X \rightarrow Y$ be a map of FMPs.

If u induces an equivalence of tangent complexes $X(E_\alpha) \rightarrow Y(E_\alpha)$ then u is an equivalence.

Remember: This was a first important step used in the proof!

What replaces



Answer: a Koszul duality type result!

Axiomatize

this will be the ingredient specific to the given context!

Def'n A weak deformation theory for $(A, \{E_\alpha\}_{\alpha \in T})$ is a functor

$$\mathcal{D}: A^{\text{op}} \longrightarrow B \quad \text{st}$$

$$\mathcal{D}: (\text{CAlg}^{\text{aug}})^{\text{op}} \longrightarrow \text{Lie}_k$$

(D1) B is a presentable ∞ -category

$$C_{\text{alg}}^*: \text{Lie} \longrightarrow (\text{CAlg}^{\text{aug}})^{\text{op}}$$

(D2) \mathcal{D} admits a left adjoint $\mathcal{D}': B \rightarrow A^{\text{op}}$

(D3) There is a full subcategory $B_0 \subseteq B$ of "good objects" s.t.:

$$g \xrightarrow{C^*} \mathcal{D} C^* g$$

(a) $\forall K \in B_0$, the unit map $K \xrightarrow{\sim} \mathcal{D}\mathcal{D}'K$ is an equiv.

(b) B_0 contains the initial obj. $\phi \in B$ $0 \in \text{Lie}_k$

$$\left(\begin{array}{c} \xrightarrow{(a)} \\ \xrightarrow{(b)} \end{array} \right) \phi \simeq \mathcal{D}\mathcal{D}'(\phi) \simeq \mathcal{D}(\star)$$

(c) $\forall \alpha \in T, n \geq 1 \exists K_{\alpha, n} \in B_0, \Omega^{\infty-n} E_\alpha \simeq \mathcal{D}'K_{\alpha, n}$

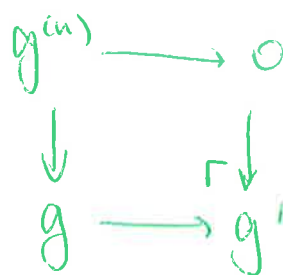
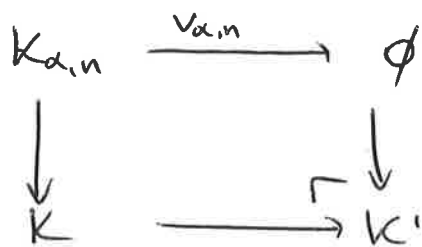
base pt of $\Omega^{\infty-n} E_\alpha$ determines

$$k \oplus k\langle L_n \rangle \simeq C^* g^{(n)}$$

$$\forall \alpha, n: K_{\alpha, n} \simeq \mathcal{D}\mathcal{D}'K_{\alpha, n} \simeq \mathcal{D}(\Omega^{\infty-n} E_\alpha) \longrightarrow \mathcal{D}(\star) \simeq \phi$$

$$\forall n: g^{(n)} \simeq \mathcal{D}(k \oplus k\langle L_n \rangle) \longrightarrow \mathcal{D}(k) \simeq 0$$

(d) For every pushout diagram



if $K \in B_0$ then $K' \in B_0$.

g good $\Rightarrow g'$ good

So, build in the main Prop we had about good Lie algebras!

Recall: had similar Prop on algebra side.

Prop: as above --

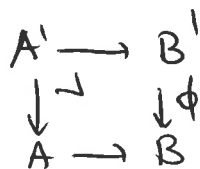
(1) \mathcal{D} carries final objects in A to initial obj. in B

(2) $A \cong \mathcal{D}'(K)$ for $K \in B_0$. Then the unit map

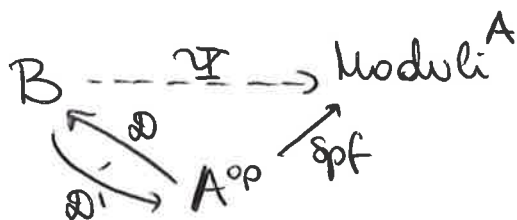
$$A \xrightarrow{\cong} \mathcal{D}'\mathcal{D}(A)$$

(3) $A \in A^{art} \Rightarrow \mathcal{D}(A) \in B_0$, $u: A \xrightarrow{\cong} \mathcal{D}'\mathcal{D}A$

(4) σ pb. $A' \rightarrow B'$ A, B Artinian, ϕ small.



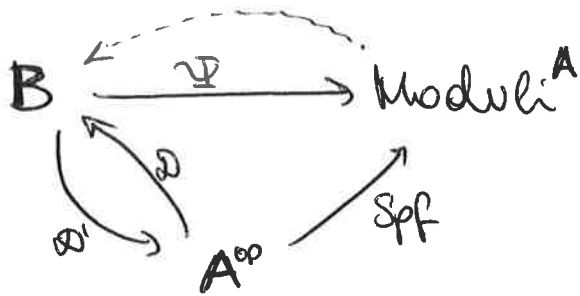
then $\mathcal{D}(\sigma)$ is p.o. in B



$$B \in B \rightsquigarrow A^{art} \in A \xrightarrow{\mathcal{D}} B^{op} \xrightarrow{j(B)} \Sigma$$

is FMP





Strategy was: use adjoint functor theorem!

Need additional hypothesis to mimic proof we had
(not necessary, will be sufficient)

Def'n A deformation theory is a weak def theory $\mathcal{D}: A^{op} \rightarrow B$
st.

$$(D1) \quad \forall \alpha \in T, \text{ let } e_\alpha: \begin{cases} B \longrightarrow Sp(\xi) \\ B \longmapsto (Sp_* \xrightarrow{E_\alpha} A \xrightarrow{\mathcal{D}} B^{op} \xrightarrow{j(\alpha)} \xi) \end{cases}$$

We require that e_α preserves small sifted colims.
and reflects equivalences.

Thm Let $(A, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and
let $\mathcal{D}: A^{op} \rightarrow B$ be a deformation theory.

Then the functor $\Psi: B \rightarrow \text{Moduli}^A$ from above
is an equivalence of ∞ -categories.

Remark: $(A, \{E_\alpha\}_{\alpha \in T})$ def context. Then the def theory
is essentially unique: it is $\text{Spt}: A^{op} \rightarrow \text{Moduli}^A$
if it exists.

Existence \Leftrightarrow requirement that tangent cplx functors
 $\text{Moduli}^A \longrightarrow Sp$ commutes w/ sifted colims

Back to motivating example:

what about associative dg algebras?

deformation context

$$A = \text{Alg}_k^{\text{aug}}$$

(arbitrary char)

$$\begin{array}{ccc} \# Sp(A) & \simeq & \text{Mod}_k \\ \downarrow & & \downarrow \\ E & \longleftarrow & k \end{array}$$

$$\Omega^{\infty-n} E \simeq k \oplus k[n]$$

Prop: $A \in \text{Alg}_k^{\text{aug}}$ is Artinian iff

- (a) A is connective: $\pi_i A \simeq 0, i < 0$ ✓
- (b) A is truncated: $\pi_i A \simeq 0, i \gg 0$ ✓
- (c) $\pi_i A$ are finite dimensional as v.s over k ✓

(d) \mathfrak{m} radical of $\pi_0(A)$
 $\xrightarrow{\text{p.d. associative algebra}/k}$
 $\Rightarrow k \rightarrow \pi_0 A / \mathfrak{m}$ is an iso.

$\pi_0 A$ local w/ nil idel \mathfrak{m}
 $k \cong \pi_0 A / \mathfrak{m}$

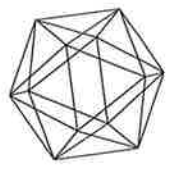
Prop: k field, $A \xrightarrow{f} B$ morphism in $\text{Alg}_k^{\text{art}}$
 f small $\Leftrightarrow \pi_0 A \xrightarrow{\pi_0 f} \pi_0 B$ surjection

Formal moduli problems Moduli⁽¹⁾

Koszul duality functor: we have seen this!

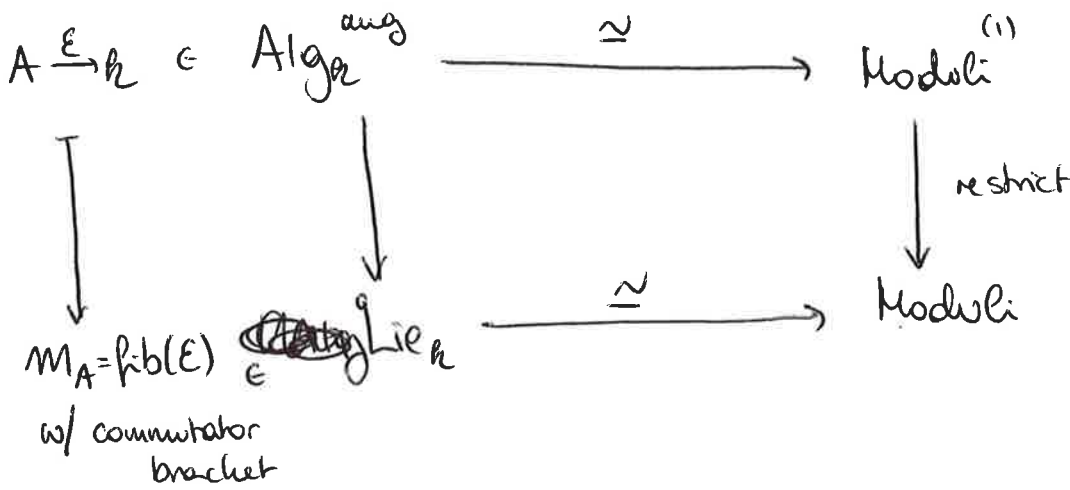
$$\mathcal{D}^{(1)} : \begin{cases} (\text{Alg}_k^{\text{aug}})^{\text{op}} & \longrightarrow & \text{Alg}_k^{\text{aug}} \\ A & \longmapsto & \text{Hom}_A(k, k) \simeq \text{Hom}_k(k \otimes_A k, k) \simeq (k \otimes_A k)^\vee \end{cases}$$

Thm: is deformation theory.



Thm $Alge_r^{aug} \xrightarrow{\cong} Moduli^{(1)}$

Recall, finally,



What are the "good" augmented algebras?

good Lie alg: $B_0 =$ coconnective and locally finite algebras

- 1. cofibrant, $\exists V \subset \mathfrak{g}_*$ st
- 2. $V_n = 0 \quad n \geq 0$
- 3. \mathfrak{g}_* free Lie alg. on V

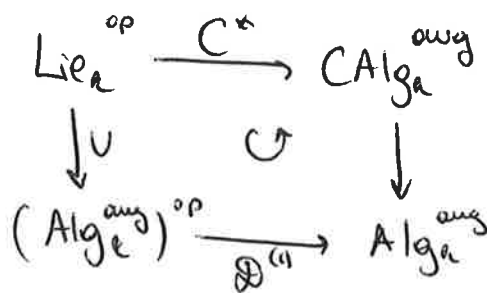
$\pi_0 A$ 1-dim'l v.sp./ \mathbb{R} gen by 1

$\pi_n A \cong 0, \quad n > 0$

$\pi_n A$ f.d. v.sp./ \mathbb{R}

B_0 was more (as prop 4.34 in Pösch)

comes from



Note: $A = U\mathfrak{g}$, have seen/discussed: $(U\mathfrak{g})^! \cong C^* \mathfrak{g}$
 $\mathcal{D}^{(1)}(U\mathfrak{g})$