Generalizations of $\mathrm{Lie}_k \underset{\Psi}{\overset{\simeq}{\rightleftarrows}} \mathrm{Moduli}$

Outlook on much more general statement

Proof of above → purely categorical part
→ certain statements specific to Com–Lie case

$\boxed{\text{Today:}}$ sketch of general statement + associative case.

Example: $A \in \mathrm{cdga}_k^{\leq 0}$.

Def'n: M a projective A-module of rank n if

(1) $\pi_0 M$ is proj. $\pi_0 A$-module of rank n

(2) M is flat over A, i.e. $\forall k$,

$$\mathrm{Tor}_0^{\pi_0 A}(\pi_k A, \pi_0 M) \xrightarrow{\simeq} \pi_k M \text{ is isom.}$$

$X(A) \subseteq \mathrm{Mod}_A$ ← ∞-category of A-modules

objects = proj. rk n A-modules
mor = equivalences } (essentially small) Kan complex

$\rightsquigarrow$ $X : \mathrm{cdga}_k^{\leq 0} \longrightarrow \mathcal{S}$

Pick $\eta \in X(k) \longleftrightarrow V$ n-dim'l v.sp./k

$X^\wedge(A) = X(A) \times_{X(k)} \{\eta\}$ = classifying space of pairs (M, α):
M proj of rk n,
$\alpha : k \underset{A}{\otimes} M \simeq V$.

$X^\wedge : \mathrm{CAlg}_k^{sm} \longrightarrow \mathcal{S}$

is a formal moduli problem.

$$\Omega^\infty T_{X^\wedge} = X^\wedge\left(k[\varepsilon]/(\varepsilon^2)\right) \simeq BG$$

$1 + \varepsilon \cdot \mathrm{End}(V) \cong G =$ Automorphisms of $k[\varepsilon]/\varepsilon^2 \otimes V$

$\equiv \mathrm{id}_V \mod \varepsilon$

$\Downarrow$

$1_V + \varepsilon\alpha \ , \ \alpha \in \mathrm{End}(V)$

$\Longrightarrow$ homology of $\mathfrak{g}.$ $\cong \begin{cases} \mathrm{End}(V) & \deg 0 = * \\ 0 & * > 0 \end{cases}$

$H_*(\mathfrak{g})$

$* < 0$: $X^\wedge(k \oplus k[d])$ ← spectrum of spaces T_X

$\wr\wr$

$\left\{ k[x_{-d}]/(x_{-d})^2 \right\}$ x_{-d} deg $-d$

spectrum of k-modules

<u>Fact</u>: any basis of V can be lifted to one for any $k[x_{-d}]/(x_{-d})^2$ - module M deforming V.

$\Rightarrow M \cong V \otimes k[\varepsilon_{-d}]/(\varepsilon_{-d})^2$

$\Longrightarrow X^\wedge(\quad)$ has only one isom. class and is connected as a space.

$\Longrightarrow H_*(\mathfrak{g}.) = 0$ for $* < 0$.

$\Longrightarrow \mathfrak{g}. \underset{q.iso}{\simeq} \mathfrak{g}$ ordinary Lie alg. $/k$

<u>Claim</u>: $\boxed{\mathfrak{g} \cong \mathrm{End}(V)}$ as Lie alg.

Q: What extra property does this have?

A: It's the Lie algebra coming from an augmented associative algebra using commutator bracket.

This will be a general pattern!

char $k=0$.

$$\begin{tikzcd}
\varepsilon: A \to k \arrow[d, mapsto] & \mathrm{Alg}_k^{aug} \arrow[r, dashed, "\simeq"] \arrow[d] & \mathrm{Moduli}^{ass} \arrow[d, "\text{restrict}"] \\
\mathfrak{m}_A & \mathrm{Lie}_k \arrow[r, "\simeq"] & \mathrm{Moduli}
\end{tikzcd}$$

$\mathfrak{m}_A$ $=$ $\mathrm{fib}(\varepsilon)$ w/ commutator bracket.

Q: • Can we generalize even more?
• Which parts of the proof work in general, which ones are specific to the context we are in?

Lurie proves a very, very general version of which the above are special examples!

Thm (sketch): Let A be an ∞-category of some algebraic objects, let $A^{art} \subseteq A$ be a full subcategory of "Artinian"/"small" objects, let $\mathrm{Moduli}^A \subset \mathrm{Fun}(A^{art}, \mathcal{S})$ be the ∞-cat of $X: A^{art} \xrightarrow[*]{} \mathcal{S}$ satisfying a certain gluing condition.

$\mathfrak{D}: A^{op} \to B$ is a "deformation theory"

Then there is an equivalence of ∞-categories of the form

$$\mathrm{Moduli}^A \simeq B$$

(*)

Let ~~Ar~~ B be a certain ∞-cat. of algebraic objects governing deformations of A-obj.

Reformulation of $X^\wedge$:

If $A \in CAlg_k^{sm}$, M connective A-module (i.e $M^i = 0$ $i>0$)

st. $M \otimes_A k \cong V$

$\Longrightarrow$ M is projective of rank n.

$$\Longrightarrow \quad X^\wedge(A) = LMod_A^{conn} \underset{LMod_k^{conn}}{\times} \{V\}$$

does $\underline{\underline{\text{not}}}$ use commutativity of A!

Can

Extend: $X_+^\wedge : Alg_k^{art} \longrightarrow \mathcal{S}$

∞-category of "Artinian" / "small" associative / "E_1-" dg algebras $/k$

Question? Is this reflected in the ~~tangent~~ $\underbrace{\text{Lie algebra}}_{\text{~~complex~~ / associated}}$? $\mathfrak{g} = \Phi(X^\wedge)$

Computation of $T_{X^\wedge}$ tangent complex:

$X^\wedge(k[\varepsilon]/(\varepsilon^2))$ gpd of order 1 deformations of the v.sp. V:

projective $k[\varepsilon]/(\varepsilon^2)$ -modules M w/ iso $M/\varepsilon M \cong V$.

Fact: any basis for V can be lifted to one of M.

$$\Longrightarrow M \cong k[\varepsilon]/(\varepsilon^2) \otimes_k V$$

morphisms in the gpd:

autom of $k[\varepsilon]/(\varepsilon^2) \otimes_k V$ which are identity modulo ε. $\overset{\text{uniquely}}{=} 1_V + \varepsilon \cdot \alpha$ $\quad \alpha \in End(V)$

General setup:

in these lectures so far:

$\mathcal{A} = \mathrm{CAlg}_k^{\mathrm{aug}} = \mathrm{CAlg}_{k/k}$

$\mathrm{Mod}_k \ni k$

$\mathrm{Sp}(\mathcal{A}) \ni \{k \oplus k[d]\}_{d \in \mathbb{N}}$, $\quad k \oplus k[d] \cong k[x_{-d}]/(x_{-d})^2$

was a spectrum E

- $\mathcal{A}$ presentable ∞-category ($\Rightarrow \exists$ final obj. $*$)
- $\mathrm{Sp}(\mathcal{A}) =:$ ∞-cat of spectrum objects

$\quad \ni F : \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{A}$ · reduced + excisive

Think: $\mathrm{Sp}(\mathcal{A})$ is holim of

$$\cdots \longrightarrow \mathcal{A}_* \xrightarrow{\Omega} \mathcal{A}_*$$

$\mathcal{A}_*$: ∞-cat of ptd obj in $\mathcal{A}$

$\Omega^{\infty-n}E = k \oplus k[n]$

$\Omega^{\infty-n} : \mathrm{Sp}(\mathcal{A}) \longrightarrow \mathcal{A}$, $\quad \Omega^{\infty-n}(X) = \Omega^m X(S^{n+m})$ for any $m \geq -n$

Definition: A <u>deformation context</u> is a pair $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$

where
- $\mathcal{A}$ is a presentable ∞-cat
- $\{E_\alpha\}_{\alpha \in T}$ is a set of objects in $\mathrm{Sp}(\mathcal{A})$

Ex: $\left(\mathrm{CAlg}_k^{\mathrm{aug}}, \{k \oplus k[d]\}\right)$, $\quad \{k \oplus k[d]\} = E$

Definition: $\phi : A' \longrightarrow A$ in $\mathcal{A}$ is <u>elementary</u> if $\exists\, \alpha \in T$, integer $n > 0$,

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, "\phi_0"] \\ A \arrow[r] & \Omega^{\infty-n} E_\alpha \end{tikzcd}$$

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & k \arrow[d] \\ A \arrow[r] & k \oplus k[n] \end{tikzcd}$$

$\phi_0 \longleftrightarrow$ image of E_α under $\mathrm{Sp}(\mathcal{A}) \xrightarrow{\Omega^{\infty-n}} \mathcal{A}$

Def'n: Let $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ def context.

$\phi : A' \longrightarrow A$ is <u>small</u> if it is finite composition of elem.

$$A' \simeq A_0 \to A_1 \longrightarrow \cdots \to A_n \simeq A$$

A is <u>Artinian</u> (small) if $A \longrightarrow *$ is small
(det. up to hom.)

$\mathcal{A}^{art} \subset \mathcal{A}$ full subcategory of Artinian obj.

CAlg_k^{sm} CCAlg_k

<u>Example</u>: $\Omega^{\infty-n} E_\alpha$ is small: $k \oplus k[n]$ is small

$$\begin{tikzcd} \Omega^{\infty-n} E_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & \Omega^{\infty-n-1} E_\alpha \end{tikzcd}$$

~~Prop~~ <u>Def'n</u> A <u>formal moduli problem</u> for $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ is a functor $X : \mathcal{A}^{art} \longrightarrow \mathcal{S}$ s.t.

(1) $X(*)$ ~~...~~ is contractible

(2) If σ:
$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$
in $\mathcal{A}^{art}$ and ϕ is small

then
$$\begin{tikzcd} X(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(B') \arrow[d] \\ X(A) \arrow[r] & X(B) \end{tikzcd}$$
in $\mathcal{S}$.

$\mathrm{Moduli}^{\mathcal{A}} \subset \mathrm{Fun}(\mathcal{A}^{art}, \mathcal{S})$ spanned by FMP's.

... ∞-cat of FMP's.

<u>Remark:</u> Have similar proposition to previously which allows to test only for

- ϕ is elementary

or even – $\phi : * \longrightarrow \Omega^{\infty-n} E_\alpha$

<u>formal spectrum</u> $A \in \mathcal{A}$. $Spf(A) : \mathcal{A}^{art} \longrightarrow \mathcal{S}$ $\in$ FMP

$$Spf(A)(B) = Map_{\mathcal{A}}(A, B)$$

$$\rightsquigarrow \quad Spf : \mathcal{A}^{op} \longrightarrow Moduli^{\mathcal{A}}$$

<u>Recall our diagram:</u>

$$\begin{tikzcd}
Lie_k \arrow[rr, bend left, "\Psi"] \arrow[dr, bend right, "C^*_{CE}"'] \arrow[ddr, bend right, "forget"'] & & Moduli \arrow[ll, bend right, "\Phi"'] \arrow[ddl, bend left, "T_x[-1]"] \\
& (CAlg_k^{aug})^{op} \arrow[ul] \arrow[ur, "Spf"'] & \\
& Mod_k &
\end{tikzcd}$$

$$\begin{tikzcd}
? \arrow[rr, dashed, bend left] \arrow[dr, dashed, bend right] & & Moduli^{\mathcal{A}} \arrow[ll, dashed, bend right] \arrow[ddl, dashed, "T"] \\
& \mathcal{A}^{op} \arrow[ul] \arrow[ur, "Spf"'] & \\
& Sp(\mathcal{A}) &
\end{tikzcd}$$

<u>Tangent space</u> of $Y : \mathcal{A}^{art} \longrightarrow \mathcal{S}$ FMP at $\alpha \in T$ is

$$Y(\Omega^\infty E_\alpha)$$

... has structure of a spectrum (of spaces):

$$S_*^{fin} \xrightarrow{E_\alpha} \mathcal{A}^{art} \xrightarrow{Y} \mathcal{S}$$ "<u>tangent complex</u> to Y at α"

We get similar proposition to previously, in this abstract setting

<u>Prop</u>: Let $u: X \to Y$ be a map of FMPs.
If u induces an equivalence of tangent complexes $X(E_\alpha) \to Y(E_\alpha)$
then u is an equivalence.

<u>Remember</u>: This was a first important step used in the proof!

What replaces $\mathrm{Lie} \rightleftarrows (\mathrm{CAlg}^{\mathrm{aug}})^{op}$?

<u>Answer</u>: a Koszul duality type result! Axiomatize this

will be the ingredient specific to the given context!

<u>Def'n</u> A <u>weak deformation theory</u> for $(A, \{E_\alpha\}_{\alpha\in T})$ is a functor

$$\mathfrak{D}: A^{op} \longrightarrow B \quad \text{st}$$

$\mathfrak{D}: (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \longrightarrow \mathrm{Lie}_k$

(D1) B is a presentable ∞-category

$C^*_{CE}: \mathrm{Lie} \longrightarrow (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$

(D2) $\mathfrak{D}$ admits a left adjoint $\mathfrak{D}': B \to A^{op}$

good Lie algebras

(D3) There is a full subcategory $B_0 \subseteq B$ of "good objects" sat:

$\mathfrak{g} \xrightarrow{\sim} \mathfrak{D}C^*\mathfrak{g}$

(a) $\forall$ $K \in B_0$, the unit map $K \xrightarrow{\sim} \mathfrak{D}\mathfrak{D}'K$ is an equiv.

(b) B_0 contains the initial obj. $\phi \in B$ $\quad 0 \in \mathrm{Lie}_k$

$$\left(\overset{(a)}{\Longrightarrow} \phi \simeq \mathfrak{D}\mathfrak{D}'(\phi) \simeq \mathfrak{D}(*) \right)$$

$\mathfrak{g}^{(n)} = \mathrm{Free}_{\mathrm{Lie}}(k[-n-1])$

(c) $\forall \alpha \in T$, $n \geq 1$ $\exists K_{\alpha,n} \in B_0$, $\Omega^{\infty-n}E_\alpha \simeq \mathfrak{D}'K_{\alpha,n}$

$k \oplus k[n] \simeq C^*\mathfrak{g}^{(n)}$

base pt of $\Omega^{\infty-n}E_\alpha$ determines

$$v_{\alpha,n}: K_{\alpha,n} \simeq \mathfrak{D}\mathfrak{D}'K_{\alpha,n} \simeq \mathfrak{D}(\Omega^{\infty-n}E_\alpha) \longrightarrow \mathfrak{D}(*) \simeq \phi$$

$$v_n: \mathfrak{g}^{(n)} \simeq \mathfrak{D}(k \oplus k[n]) \longrightarrow \mathfrak{D}(k) \simeq 0$$

(d) For every pushout diagram

$$\begin{tikzcd} K_{\alpha,n} \arrow[r, "v_{\alpha,n}"] \arrow[d] & \emptyset \arrow[d] \\ K \arrow[r] & K' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

if $K \in \mathbf{B}_0$ then $K' \in \mathbf{B}_0$.

$$\begin{tikzcd} \mathfrak{g}^{(n)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{g}' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\mathfrak{g}$ good $\Rightarrow$ $\mathfrak{g}'$ good

So, build in the main Prop we had about good Lie algebras!

insert (*) here

Recall: had similar Prop on algebra side.

Prop: As above ..

(1) $\mathfrak{D}$ carries final objects in $\mathbf{A}$ to initial obj. in $\mathbf{B}$

$\mathfrak{D}(k) \simeq 0$

(2) $A \simeq \mathfrak{D}'(K)$ for $K \in \mathbf{B}_0$. Then the unit map

$$A \xrightarrow{\simeq} \mathfrak{D}'\mathfrak{D}(A)$$

(3) $A \in \mathbf{A}^{art} \Rightarrow \mathfrak{D}(A) \in \mathbf{B}_0$, $u: A \xrightarrow{\simeq} \mathfrak{D}'\mathfrak{D}A$

(4) σ p.b.

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

A, B Artinian, ϕ small.

Then $\mathfrak{D}(\sigma)$ is p.o. in $\mathbf{B}$

$$\begin{tikzcd} \mathbf{B} \arrow[rr, dashed, "\Psi"] \arrow[dr, bend right, "\mathfrak{D}'"'] & & \mathrm{Moduli}^{A} \\ & \mathbf{A}^{op} \arrow[ul, "\mathfrak{D}"] \arrow[ur, "\mathrm{Spf}"'] & \end{tikzcd}$$

$B \in \mathbf{B} \rightsquigarrow$

$$\mathbf{A}^{art} \subseteq \mathbf{A} \xrightarrow{\mathfrak{D}} \mathbf{B}^{op} \xrightarrow{j(B)} \mathcal{S}$$

is FMP

$$\begin{tikzcd}
\mathbf{B} \arrow[rr, "\Psi"] \arrow[dr, bend right, "\mathfrak{D}'"'] & & \mathrm{Moduli}^{\mathbf{A}} \arrow[ll, bend right, dashed] \\
& \mathbf{A}^{op} \arrow[ul, "\mathfrak{D}"'] \arrow[ur, "\mathrm{Spf}"'] &
\end{tikzcd}$$

<u>Strategy was:</u> use adjoint functor theorem!

Need additional hypothesis to mimic proof we had
(not necessary, will be sufficient)

<u>Def'n</u> A <u>deformation theory</u> is a weak def theory $\mathfrak{D}: \mathbf{A}^{op} \to \mathbf{B}$ st.

(D1) $\forall \alpha \in T$, let $e_\alpha : \begin{cases} \mathbf{B} \longrightarrow \mathrm{Sp}(\mathcal{S}). \\ B \longmapsto \left(\mathcal{S}_*^{fin} \xrightarrow{E_\alpha} \mathbf{A} \xrightarrow{\mathfrak{D}} \mathbf{B}^{op} \xrightarrow{j(K)} \mathcal{S} \right) \end{cases}$

We require that e_α preserves small sifted colims. and reflects equivalences.

<u>Thm</u> Let $(\mathbf{A}, \{E_\alpha\}_{\alpha \in T})$ be a deformation context and let $\mathfrak{D}: \mathbf{A}^{op} \to \mathbf{B}$ be a deformation theory.
Then the functor $\Psi: \mathbf{B} \to \mathrm{Moduli}^{\mathbf{A}}$ from above is an equivalence of ∞-categories.

<u>Remark:</u> $(A, \{E_\alpha\}_{\alpha \in T})$ def context. Then the def theory is essentially unique: it is $\mathrm{Spf}: \mathbf{A}^{op} \to \mathrm{Moduli}^{\mathbf{A}}$ if it exists.
Existence $\Leftrightarrow$ requirement that tangent cplx functors $\mathrm{Moduli}^{\mathbf{A}} \longrightarrow \mathrm{Sp}$ commutes w/ sifted colims

Back to motivating example:

what about <u>associative</u> dg algebras?

<u>deformation context</u>

$\mathbf{A} = \mathrm{Alg}_k^{aug}$ (arbitrary char)

$\mathrm{Sp}(\mathbf{A}) \simeq \mathrm{Mod}_k$

$$\begin{array}{ccc} \Psi & & \Psi \\ E & \longleftarrow & k \end{array}$$

$\Omega^{\infty-n} E \simeq k \oplus k[n]$

<u>Prop:</u> $A \in \mathrm{Alg}_k^{aug}$ is Artinian iff

(a) A is connective: $\pi_i A \simeq 0$, $i < 0$ ✓

(b) A is truncated: $\pi_i A \simeq 0$, $i >> 0$ ✓

(c) $\pi_i A$ are finite dimensional as v.s over k ✓

(d) $\mathfrak{m}$ radical of $\pi_0(A)$ ← f.d. associative algebra/k

$\Rightarrow k \longrightarrow \pi_0 A/\mathfrak{m}$ is an iso.

$\pi_0 A$ local w/ max'l ideal $\mathfrak{m}$
$k \xrightarrow{\simeq} \pi_0 A/\mathfrak{m}$

(<u>Prop:</u> k field, $A \xrightarrow{f} B$ morphism in Alg_k^{art}

f small $\Leftrightarrow$ $\pi_0 A \xrightarrow{\pi_0 f} \pi_0 B$ surjection)

$\rightsquigarrow$ Formal moduli problems $\mathrm{Moduli}^{(1)}$

<u>Koszul duality functor</u>: we have seen this!

$$\mathfrak{D}^{(1)} : \begin{cases} (\mathrm{Alg}_k^{aug})^{op} \longrightarrow \mathrm{Alg}_k^{aug} \\ A \longmapsto \mathrm{Hom}_A(k,k) \simeq \mathrm{Hom}_k(k \otimes_A k, k) \simeq (k \otimes_A k)^{\vee} \end{cases}$$

<u>Thm:</u> is deformation theory.

Thm $\mathrm{Alg}_k^{aug} \xrightarrow{\simeq} \mathrm{Moduli}^{(1)}$

Recall, finally,

$$\begin{tikzcd}
A \xrightarrow{\varepsilon} k \in \mathrm{Alg}_k^{aug} \arrow[rr, "\simeq"] \arrow[d, mapsto] \arrow[d] & & \mathrm{Moduli}^{(1)} \arrow[d, "\text{restrict}"] \\
m_A = \mathrm{fib}(\varepsilon) \in \mathrm{Lie}_k \arrow[rr, "\simeq"] & & \mathrm{Moduli}
\end{tikzcd}$$

w/ commutator bracket

What are the "good" augmented algebras?

$B_0 =$ coconnective and locally finite algebras

$\pi_0 A$ 1-dim'l v.sp. $/k$ gen by 1

$\pi_n A \simeq 0$, $n > 0$

$\pi_n A$ f.d. v.sp $/k$

<u>good Lie alg:</u>

cofibrant, $\exists V \subset g_*$ st

1. V_n fin. dim'l
2. $V_n = 0$ $n \geq 0$
3. g_* free Lie alg. on V

B_0 was more ($\rightsquigarrow$ prop 4.3.4, in Porta)

Comes from

$$\begin{tikzcd}
\mathrm{Lie}_k^{op} \arrow[r, "C^*"] \arrow[d, "U"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{CAlg}_k^{aug} \arrow[d] \\
(\mathrm{Alg}_k^{aug})^{op} \arrow[r, "\mathcal{D}^{(1)}"'] & \mathrm{Alg}_k^{aug}
\end{tikzcd}$$

<u>Note:</u> $A = \mathcal{U}g$, have seen/discussed: $(\mathcal{U}g)^! \simeq C^* g$

$\mathcal{D}^{(1)}(\mathcal{U}g)$